Characterization of Unique Boundary Point Graphs

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Abstract: For a connected graph G, the distance between two points is the length of a shortest path joining them. The eccentricity $e(u)$ of a point u is the distance to a point farthest from u. A point v is a boundary point of a point u if $d(u,w) \leq d(u,v)$ for all $w \epsilon N(v)$.. A graph G is said to be unique boundary point graph if every point of G has a unique boundary point. The NazziSchnedermann style of proof is followed in characterizing the family of graphs which is a Unique Boundary point graphs or not.

Keywords: Graph, Point degree, Point distant,Boundary Point, Unique Boundary Point graphs.

1. Introduction

The concept of eccentricity was generated with the introduction of the concept of boundary point by Chartrand [1]. A pointv is called a boundary point of u if $d(u,w) \leq d(u,v)$ for all $w \epsilon N(v)$. A subtle difference between eccentric point and a boundary point is that the above inequality holds good without being an eccentric point. That is, consider the graph shown in Figure 1.

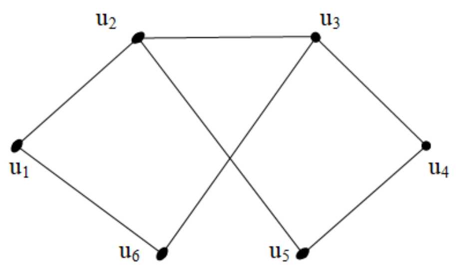

Figure 1

Note that the pointu_4 is the eccentric point of u_1 and u_1 is the eccentric point of u_1 as $e(u_1) = e(u_4) = 3$. Clearly $d(u_j, u_1) \leq d(u_1, u_4) \forall j \in N(u_4)$.Observe that u_5 is the boundary point of u1 as $d(u_j, u_1) \leq d(u_1, u_5) \forall c$ this is because $d(u_1, u_2) = 1 \leq d(u_1, u_5) = 2$ and $d(u_1, u_6) = 1 \leq d(u_1, u_5) = 2$ where $N(u_5) = \{u_1, u_6\}$. Notice have that u_5 is not the eccentric point of u_1.
A point $u \in V(G)$ is said to be complete if the subgraph induced by the neighbors of u is a complete graph. A graph G is called the unique eccentric point graph, if every point of G has exactly one eccentric point. In [5]the concept of a unique eccentric point graph was first developed byParthasarathy and Nandhakumar. A graph G is referred to as a unique eccentric point graph if each point contains exactly one eccentric point. The pathways P_{2n} on an even number of points form a basic class of unique eccentric point graphs. Theoretical work on even graphs was done by [3]. Even in the case that there exists a unique point, the friend of u, such that $d(u_i, u_j) = \text{diam}\, G$, for every pointu of G, the graph is considered nontrivial linked. Additionally, graphs can be referred to as self-centered unique eccentric point graphs and diametrical graphs.For instance, P_{2n}, the path on 2n points is the unique eccentricpoint graph this is because, if
$P2n = u1u2 \ldots u2n$ then $e(u1) = e(u2n), e(u2) = e(u2n-1), \ldots,$
$e(u2n) = e(u2n+1-2)$.

The idea of boundary graphs was first suggested by Marimuthu and SivanandhaSaraswathy [4]. If $d(u,w) \leq d(u,v)$ for all u, then a pointv is a boundary point of a point u. Given a graph G, form another graph B*(G) called boundary graph of G by having
$V(B*(G)) = V(G)$ and two pointsu and v in $B*(G)$ are said to be adjacent if either u is a boundary of v or v is a boundary of u. Alternately, one can call a graph G, a boundary graph if there exists a graph G, a boundary graph H there exists a graph H such that $G = B*(H)$.

We call a graph G, a unique boundary point graph if every point of G has exactly one boundary point. Observe that a unique eccentric point graph need not be a unique boundary point graph. For instance,P_{2n} is the unique eccentric point graph but not a unique boundary point graph. This is because every two degreepoint has more than one boundary point.

The Steiner distance is a generalization of distance that was first described by Chartrand et al.[2] in 1989. The smallest size among all connected subgraphs whose point sets contain S is the steiner distance $d(S)$ among the points of a given set, S. Based on this, we presented the characterization of a unique boundary point graph using the steiner boundary distance, a new generalization of distance that we introduced in [5].

2. Definition and Unique Boundary Point Graphs Classification

Definition:2.1
A graph G is said to be a unique boundary point graph if every point of G has exactly one boundary point.
Proposition 2.2: The even cycle graph C_{2k} of diameter k, $(k \geq 2)$ is a Unique boundarypoint graph.
Proof:
Let$C_{2k} = u_1u_2\ldots u_k,\ldots u_{2k-1}u_{2k}u_1$. Note that $d(u1, uk+1) \geq d(w, u1)$ where $w \in N(u_{k+1})$. So in general, $d(w, ui) \leq d(ui, uk+i)$ $\forall w \in N(u_{k+i})$. That is each $uj \in V(C2k)$ has an unique boundary point$uj + k$ for $1 \leq j \leq k$ and each $uj \in V(C2k)$ has an Unique boundary point $uj - k$ for $k + 1 \leq j \leq 2k$. So C_{2k} is a unique boundary point graph.
Note: As each point has more than one boundary point in C_{2k+1}, one can deduce that C_{2k+1} is not a Unique boundary point graph.
Proposition: 2.3
The K dimensional cube Q_k of diameter k, $(k \geq 1)$ is a Unique boundary point graph.
Proof:
Let $V(Q_k) = \{u_1, u_2, ..u_{2k}\}$. As diameter $(Q_k) = k$, one can choose the peripheral point of each u_i and observe that they happen to be the Unique boundary point. That is, each $uj \in V(Qk)$ has an Unique boundarypoint u_{2^k-j+1} for $\begin{cases} 1 \leq j \leq 2^{k-1} \\ 2^{k-1}+1 \leq j \leq 2^k \end{cases}$. This means $d(w,u) \leq d(u,v)$ for all $w \in N(v)$. So Qk is a Unique boundary point graph.
Proposition:2.4
The graph $\overline{kK_2}$ is a unique boundary point graph for all $k \geq 2$.
Proof:
Let $V(\overline{kK_2})$for $k \geq 2$ be $\{(v_i, v_j) : 1 \leq j \leq k\}$. One can observe that for each $v_j \in V(\overline{kK_2})$ the point$v_j \in V(\overline{kK_2})$ is the Unique boundary point in $\overline{kK_2}$. This is because, each vjis adjacent with all pointvr, for all $1 \leq r \leq k$, except $r = j$. Hence
$d(w,u) \leq d(u,v)$ for all $w \in N(v)$. so $\overline{kK_2}$ is a unique boundary point graph for allk ≥ 2.
Note: When $k = 1$ in Proposition 3, then $\overline{kK_2} \approx 2K_1$ which is a union of disconnected empty graphs.
Proposition 2.5:
The antiprism graph A_{2n} with
$V(A_{2n}) = \{u_i, v_i : 1 \leq i \leq n\}$ and
$E(A_{2n}) = \{(u_i, u_{i+1}) : 1 \leq i \leq n(\bmod n)\} \cup \{(v_i, v_{i+1}) : 1 \leq i \leq n(\bmod n)\} \cup$.
$\{(u_i, v_i) : 1 \leq i \leq n\} \cup \{(u_i, v_{i+1}) : 1 \leq i \leq n\}$ for $n \geq 3$ is a Unique boundary pointgraph.

Proof:

Here first observe that the farthest distance of any point is $\left\lceil \frac{n}{4} \right\rceil + 1.$ Hence for each point $u \in V(A_{2n})$ there exists a unique point $u \in V(A_{2n})$ with $d(u,v) = \left\lceil \frac{n}{4} \right\rceil + 1.$

For instance, if $u_j \in V(A_{2n})$ then $v_{j+\left\lceil \frac{n}{4} \right\rceil + 1 (\mathrm{mod}\, n)} \in V(A_{2n})$ acts as the Unique boundary point and if $v_j \in V(A_{2n})$ then $u_{j+\left\lceil \frac{n}{4} \right\rceil + 1 (\mathrm{mod}\, n)} \in V(A_{2n})$ acts as the Unique boundary point. So, A_{2n} is a unique boundary point graph.

Let F_{11}, F_{12}, F_{22}, F_{23}, F_{24} and F_3 denote the set of all graphs G such that $r(G) = 1$ and $d(G) = 1$; $r(G) = 1$ and $d(G) = 2$; $r(G) = 2$ and $d(G) = 2$ $r(G) = 2$ and $d(G) = 3$; $r(G) = 2$ and $d(G) = 4$ and $r(G) \geq 3$ respectively.

We now consider the task of identifying the families of graphs that are unique boundary point graphs. In our search process, we probed first there families of graphs that possess a given radius and diameter. By F_{2j} family of graphs we mean those graphs that have radius i and diameter j. For instance, if i= 1, j = 1 then F_{11} is a complete graph. In F_{11}, every point is at a distance one with ever other point. So every point of a graph in F_{11} is a boundary point of every other point. So, no graph in F_{11} is a Unique boundary point graph. As the diameter is twice the radius and every graph in F_{12} also possesses the same characteristic as F_{11}, one can conclude that F_{12} family of graphs are also not unique boundary point graphs.

Next we consider the problem of identifying those classes of graphs in F_{22} that are

a) Unique boundary point graph

b) Not a Unique boundary point graph.

We know that if the F_{rr} family of graphs are also called Self-centered graphs. These families of graphs possess the characteristic that each point has more than one boundary point. For instance, the complete multipartite graph $K_{n,n_2,\ldots n_{k1}}$ is not aunique boundary point graph due to the fact that they belong to the family F_{22}.

Proposition 2.6:

Any graph $G \in F_{22}$ is a unique boundary point graph if and only if for each.

Proof:

Suppose that a graph $u \in V(G)$, $|\mathrm{V(G)} - \mathrm{N_G}[u]| = 1, G \in F_{22}$ is a Unique boundary point graph, By definition, it follows that every point $u \in V(G)$ has a Unique boundary point, say v. This means that every point w ≠ v $w \neq v \notin V(G)$ is at a distance one (j) with u. So, $|\mathrm{V(G)} - \mathrm{N_G}[u]| = 1$, $N_j[u] = 1$. Conversely, if $|\mathrm{V(G)} - \mathrm{N_G}[u]| = 1$, then it means that for every point $u \in V(G)$ there exists exactly one point $v \in V(G)$ at a distance two from u. So, v is a Unique boundary point of u and G is a unique boundary graph.

Proposition: 2.7

Any graph $G \in F_{rr}$, $r \geq 2$ is a Unique boundary point if and only if for each $u \in V(G)$, $\left| V(G) - \sum_{j=1}^{r} |N_j[u]| \right| = 1$ where $N_j[u] = \{v \in V(G) : d(u,v) = j\}$.

Proposition: 2.8

Q_r is a Unique boundary point graph in F_{rr}.

Proof:

We know that Q_r is a r-regular graph with $|V(Q_r)| = 2^r$ and $|(Q_r)| = r2^{r-1}$. Let $u \in V(Q_r)$ be any arbitrary point. Clearly, there is $\binom{r}{1}$ point in Q_r that are at distance 1 from u, $\binom{r}{2}$ points in Q_r that are at distance 2 from u, …, $\binom{r}{r-1}$ points in Q_r that are at distance r-1 from u. Also

$$\left|V(G) - \sum_{j=1}^{r-1} |N_j[u]|\right| = 2^r - \left\{\binom{r}{1} + \binom{r}{2} + \ldots + \binom{r}{r-1}\right\} = 2^r - \{2^r - 1\} = 1.$$

So, by Proposition 6, it follows that Q_r is a Unique boundary point graph in F_{rr} .

Proposition: 2.9

Any r-regular even graph with 2r points is a Unique boundary point graph in $F_{r,r}$.

Proof:

Let G be any r-regular, even graph with 2n points. Let $u \in V(G)$ be any arbitrary point. Note that, there is $\binom{r}{1} = r$ points in G that are at a distance 2 from u, …, $\binom{r}{2}$ points in G that are distance (r-1) from u and $\binom{r}{r-1}$ points in G that are at distance (r-1) from u and $\binom{r}{r}$ =1 point at a distance r from u. Also $\left|V(G) - \sum_{j=1}^{r-1} |N_j[u]|\right| = 2n - \left\{\binom{r}{1} + \binom{r}{2} + \ldots + \binom{r}{r-1}\right\} = 2n - \{2^r - 1\} = 2n - 2^r + 1.$

Clearly it leads to the result that any r-regular even graph with 2^r points is a Unique boundary point graph in $F_{r,r}$.

Lemma: 2.10

If $G \in F_{24}$ then G is not an unique boundary point graph.

Proof:

Consider a graph $G \in F_{24}$. Let v be a central point in G and let u and w be the two antipodal points in G. Now d(u,v) = d(w,v) = r(G). This implies the point v has two eccentric points and they are boundary points of v. Hence G is not an unique boundary point graph.

If $G \in F_3$, then it is not necessary that the graph G to be an unique boundary point graph. For example, the graph C_9 is not a unique boundary point graph.

3. Conclusion

In a previously published research study, we introduced the Steiner boundary distance - a generalization of distance. In this instance, we constructed "unique boundary point graphs" and investigated the presence of these graphs by establishing a relationship between the ordinary and Steiner boundary distances. In further we can apply the concept in real life applications like terrain mapping and surveying in digital projects.

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