A Mathematical Model on Women's Osteoporosis

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Abstract: In this paper, a mathematical model of women's osteoporosis is formulated. Three control parameters used such as diet with exercise, nutrition intake and medication. The equilibrium analysis is performed. The boundedness and positiveness of the osteoporosis models are performed. The local and global stability is studied using the R.H. criteria and Lyapunov's approach. The reproduction number of osteoporosis is calculated. The sensitivity analysis is performed for the model. Numerical simulations are performed to show the flow of the variables using MATLAB.

Keywords: Osteoporosis, Positiveness and boundedness, Basic Reproduction Number, Sensitivity Analysis, Stability.

1. Introduction

Osteoporosis is a medical condition resulting in the fragility of bones, diminution in bone density vulnerability to fractures. Commonly known as "silent disease" affects older people, postmenopausal women and also predominantly affects people with hormonal changes, men and younger individuals. Osteoporosis grows in human body when loses too much bone or has lesser growth in bone development. Among women osteoporosis wide spread due to biological, environmental and hormonal factors that affects their bone density. About 30 percent of postmenopausal women are affected by osteoporosis globally. The risk of being affected by osteoporosis increases considerably after fifty years. The women most at risk are those with low BMI, chronic illness, a family history of Osteoporosis or bone fracture, and declined estrogen levels.

Comparatively 80 percentage of women are affected with osteoporosis. Common risk factors for osteoporosis in women are bone structure, longevity, menopause, hormonal fluctuations, pregnancy and breast feeding. Preventive measures for osteoporosis are sufficient intake of calcium, healthy diet, regular exercise to maintain bone density, eventual screening for bone density after menopause and intake of medication.

Many authors illustrated a mathematical model of osteoporosis. Graham J.M. [1] developed of a dynamical model on the role of osteocytes in targeted bone remodeling. Komarova S.V. [5] discussed a mathematical model for autocrine osteocalast regulation in control of bone remodeling. Scheiner S. [7] formulated a mathematical model of postmenopausal osteoporosis and used treatment with the anticatabolic drug denosumab. Amirouche F. [1] studied a mathematical model of an approach for biochemical processes of bone remodeling. Cholamjiak W. [2] evaluated a mathematical model for bone transformation with intelligent artificial investigations. Cook C.V. [3] illustrated a dynamical analysis on bone remodeling from a biology perspective. Pivonka P. [6] introduced a mathematical model on structure and control of bone remodeling. Siewe N. [8] introduced a mathematical model of osteoporosis induced by cellular senescence. In this study, we have developed a mathematical model of osteoporosis in women and introduced three control variables to the diet with exercise, nutrition intake, and medication.

2. Equations of Osteoporosis model

The following system of ODE symbolizes the Osteoporosis model:

$$\frac{dS_W}{dt} = \Omega N_W - (\sigma O_1 + r_1 + r_2 + \mu_w) S_W$$

$$\left.\begin{aligned}\frac{\mathrm{dO_0}}{\mathrm{dt}} &= \sigma \mathrm{S_W} O_1 - (\delta + r_1 + r_3 + \mu_\mathrm{w})\mathrm{O_0} \\ \frac{\mathrm{d}O_1}{\mathrm{dt}} &= \delta O_0 - (\gamma_\mathrm{W} + r_3 + \mu_\mathrm{w})O_1 \\ \frac{\mathrm{dR_W}}{\mathrm{dt}} &= (r_1 + r_2)S_\mathrm{W} + (r_1 + r_3)\mathrm{O_0} + (\gamma_\mathrm{W} + r_3) - \mu_\mathrm{w}\mathrm{R_W}\end{aligned}\right\} \quad (1)$$

$$\mathrm{N_W}\ (t) = \mathrm{S_W}(t) + \mathrm{O_0}(t) + \mathrm{O_1}(t) + \mathrm{R_W}(t)$$

With $\mathrm{S_W}(t)$, $\mathrm{O_0}(t)$, $\mathrm{O_1}(t)$, $\mathrm{R_W}(t) \geq 0$.

Also $\mu_\mathrm{w}, \Omega, \delta, \mathrm{r_1}, \mathrm{r_2}, \mathrm{r_3}, \gamma_\mathrm{W} > 0$.

$\mathrm{N_M}\ (t) = \mathrm{S_P}(t) + \mathrm{I_P}(t) + \mathrm{R_P}(t)$,

$\mathrm{N_W}\ (t) = \mathrm{S_W}(t) + \mathrm{O_0}(t) + \mathrm{O_1}(t) + \mathrm{R_W}(t)$

Where $\mathrm{S_W}(t)$ - Susceptible, $\mathrm{O_0}(t)$ - Osteopenia, $\mathrm{O_1}(t)$ - Osteoporosis, $\mathrm{R_W}(t)$ - Recovered.

Ω- Mean natality rate of female, μ_w- Mean mortality rate of female, γ_W - Women recovery rate, σ - Transition rate, δ - Exposed rate, $\mathrm{N_W}$ - Female Population.

r_1, r_2, r_3 are the control parameters such as diet with exercise, nutrition intake and medication.
The Osteoporosis model is given below:

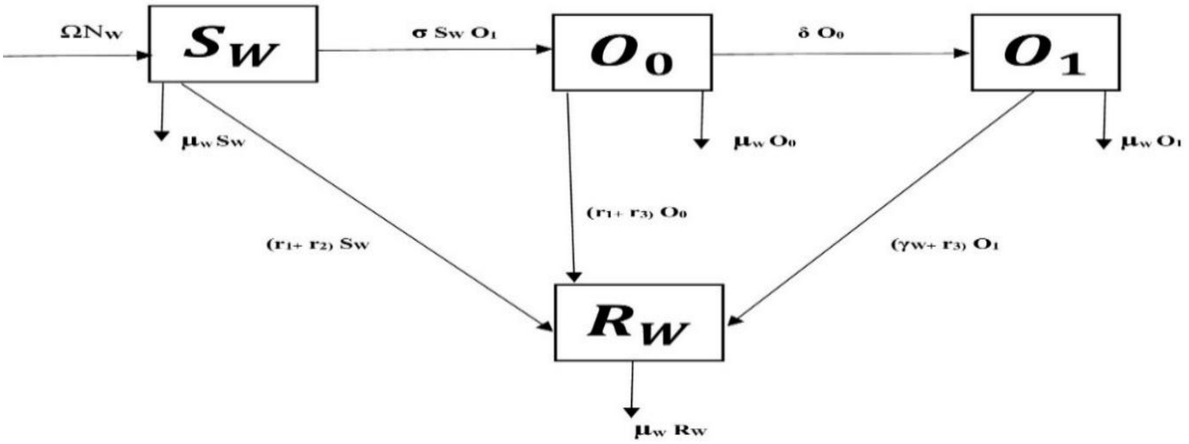


Figure 1: Osteoporosis model

3. Boundedness and Positiveness

From (1)

$$\frac{\mathrm{dS_W}}{\mathrm{dt}} \geq -(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W} \quad (2)$$

$$\frac{\mathrm{dO_0}}{\mathrm{dt}} \geq -(\delta + r_1 + r_3 + \mu_\mathrm{w})\mathrm{O_0} \quad (3)$$

$$\frac{\mathrm{d}O_1}{\mathrm{dt}} \geq -(\gamma_\mathrm{W} + r_3 + \mu_\mathrm{w})O_1 \quad (4)$$

$$\frac{\mathrm{dR_W}}{\mathrm{dt}} \geq -\mu_\mathrm{w}\mathrm{R_W} \quad (5)$$

From (2),

$$\frac{1}{\mathrm{S_W}}\mathrm{dS_W} \geq -(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W} dt$$

Integrating,

$$\int_0^t \frac{1}{\mathrm{S_W}}\mathrm{dS_W} \geq -(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W} \int_0^t \mathrm{dt}$$

$$log \mathrm{S_W}(t) - log\, \mathrm{S_W}(0) \geq -(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W}(t - 0)$$

$$log \frac{\mathrm{S_W}(t)}{\mathrm{S_W}(0)} \geq -(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W} t$$

$$\frac{\mathrm{S_W}(t)}{\mathrm{S_W}(0)} \geq e^{-(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W}\, t}$$

$$\mathrm{S_W}\ (t) \geq \mathrm{S_W}\ (0)\ e^{-(\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})S_\mathrm{W}\, t}$$

$$\mathrm{S_W}\ (t) \geq 0 \text{ as } \mathrm{S_W}\ (0) \geq 0$$

From (3),

$$\frac{1}{\mathrm{O_0}}\mathrm{dO_0} \geq -(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0}dt$$

Integrating,

$$\int_0^t \frac{1}{\mathrm{O_0}}\mathrm{dO_0} \geq -(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0} \int_0^t \mathrm{dt}$$

$$log\ \mathrm{O_0}(t) - log\ \mathrm{O_0}(0) \geq -(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0}(t - 0)$$

$$log\frac{\mathrm{O_0}(t)}{\mathrm{O_0}(0)} \geq -(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0}t$$

$$\frac{\mathrm{O_0}(t)}{\mathrm{O_0}(0)} \geq e^{-(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0}\ t}$$

$$\mathrm{O_0}\ (t) \geq \mathrm{O_0}\ (0)\ e^{--(\delta + r_1 + r_3 + \mu_w)\mathrm{O_0}t}$$

$$\mathrm{O_0}\ (t) \geq 0 \text{ as } \mathrm{O_0}\ (0) \geq 0$$

From (4),

$$\frac{1}{O_1}\mathrm{d}O_1 \geq -(\gamma_W + r_3 + \mu_w)O_1dt$$

Integrating,

$$\int_0^t \frac{1}{O_1}\mathrm{d}O_1 \geq -(\gamma_W + r_3 + \mu_w)O_1 \int_0^t \mathrm{dt}$$

$$log\ O_1(t) - log\ O_1(0) \geq -(\gamma_W + r_3 + \mu_w)O_1(t - 0)$$

$$log\frac{O_1(t)}{O_1(0)} \geq -(\gamma_W + r_3 + \mu_w)O_1t$$

$$\frac{O_1(t)}{O_1(0)} \geq e^{-(\gamma_W + r_3 + \mu_w)O_1t}$$

$$O_1\ (t) \geq O_1\ (0)\ e^{-(\gamma_W + r_3 + \mu_w)O_1t}$$

$$O_1\ (t) \geq 0 \text{ as } O_1\ (0) \geq 0$$

From (5),

$$\frac{1}{\mathrm{R_W}}\mathrm{dR_W} \geq -\mu_w\ dt$$

Integrating,

$$\int_0^t \frac{1}{\mathrm{R_W}}\mathrm{dR_W} \geq -\mu_w \int_0^t \mathrm{dt}$$

$$log\ \mathrm{R_W}(t) - log\ \mathrm{R_W}(0) \geq -\mu_w\ (t - 0)$$

$$log\frac{\mathrm{R_W}(t)}{\mathrm{R_W}(0)} \geq \mu_w\ t$$

$$\frac{\mathrm{R_W}(t)}{\mathrm{R_W}(0)} \geq e^{-\mu_w\ t}$$

$$\mathrm{R_W}\ (t) \geq \mathrm{R_W}\ (0)\ e^{-\mu_w t}$$

$$\mathrm{R_W}\ (t) \geq 0 \text{ as } \mathrm{R_W}\ (0) \geq 0$$

Hence positiveness of the model is proved.
Boundedness of the model
From (2)
$\mathrm{N_W}$ (t)= $\mathrm{S_W}$(t)+ $\mathrm{O_0}$(t)+ $\mathrm{O_1}$(t)+ $\mathrm{R_W}$(t)
Differentiating,

$$\frac{d\mathrm{N_W}}{dt} = \frac{d\mathrm{S_W}}{dt} + \frac{d\mathrm{O_0}}{dt} + \frac{d\mathrm{O_1}}{dt} + \frac{d\mathrm{R_W}}{dt}$$

$$\frac{dN_W}{dt} = (\Omega - \mu_w)N_W$$

$$\frac{1}{dt} dN_W = (\Omega - \mu_w)dt$$

Integrating,

$$\int_0^t \frac{1}{N_W} dN_W \geq \int_0^t (\Omega - \mu_w)\, dt$$

$$\Rightarrow N_W(t) \leq N_W(0)\, e^{(\Omega-\mu_w)t}$$

Hence, N_W (t) is bounded by a positive integer.
As S_W(t), O_0(t), O_1(t), R_W(t) are positive and N_W (t)= S_W(t)+ O_0(t)+ O_1(t)+ R_W(t)
We conclude that S_W(t), O_0(t), O_1(t) and R_W(t) are bounded.

4. Equilibrium Analysis

Disease free equilibrium: $G_0(\widetilde{S_W}, 0,0,0,)$.
Let $\widetilde{S_W}$ be the positive solutions of $\frac{dS_W}{dt} = 0$.
From (1),

$$\widetilde{S_W} = \frac{\Omega N_W}{r_1 + r_2 + \mu_w}$$

$$G_0(\widetilde{S_W}, 0,0,0) = \left(\frac{\Omega N_W}{r_1 + r_2 + \mu_w}, 0,0,0\right)$$

Endemic equilibrium: $G_1(S_W{}^*, O_0{}^*, O_1{}^*, R_W{}^*)$
Let the positive solutions of $\frac{dS_W}{dt} = 0, \frac{dO_0}{dt} = 0, \frac{dO_1}{dt} = 0, \frac{dR_W}{dt} = 0$ be $S_W{}^*$, $O_0{}^*$, $O_1{}^*$and $R_W{}^*$
From (1),

$$S_W{}^* = \frac{(\gamma_W + r_3 + \mu_w)(\delta + r_1 + r_3 + \mu_w)}{\sigma\delta}$$

$$O_0{}^* = \frac{1}{\delta}\left(\frac{(\Omega N_W \delta\sigma) - [(r_1 + r_2 + \mu_w)(\gamma_W + r_3 + \mu_w)(r_1 + r_3 + \mu_w)]}{(\delta + r_1 + r_3 + \mu_w)}\right)$$

$$O_1{}^* = \frac{1}{\sigma}\left(\frac{\Omega N_W \delta\sigma - [(r_1 + r_2 + \mu_w)(\gamma_W + r_3 + \mu_w)(\delta + r_1 + r_3 + \mu_w)]}{(\gamma_W + r_3 + \mu_w)(\delta + r_1 + r_3 + \mu_w)}\right)$$

$$R_W{}^* = \frac{(r_1 + r_2)(\gamma_W + r_3 + \mu_w) + (r_1 + r_2)[\Omega N_W \delta\sigma(\delta + r_1 + r_3 + \mu_w) - (r_1 + r_2 + \mu_w)] + a^*}{\mu_w \delta^2 \sigma^2 (\delta + r_1 + r_3 + \mu_w)}$$

Where $a^* = (\gamma_W + r_3)[\, \Omega N_W \delta\sigma - (\delta + r_1 + r_3 + \mu_w)(r_1 + r_2 + \mu_w)]$
∴ endemic equilibrium is given by

$$G_1(S_W{}^*, O_0{}^*, O_1{}^*, R_W{}^*) = \{\frac{(\gamma_W+r_3+\mu_w)(\delta+r_1+r_3+\mu_w)}{\sigma\delta}, \frac{1}{\delta}\left(\frac{(\Omega N_W\delta\sigma)-[(r_1+r_2+\mu_w)(\gamma_W+r_3+\mu_w)(r_1+r_3+\mu_w)]}{(\delta+r_1+r_3+\mu_w)}\right), \frac{1}{\sigma}\left(\frac{\Omega N_W\delta\sigma-[(r_1+r_2+\mu_w)(\gamma_W+r_3+\mu_w)(\delta+r_1+r_3+\mu_w)]}{(\gamma_W+r_3+\mu_w)(\delta+r_1+r_3+\mu_w)}\right), \frac{(r_1+r_2)(\gamma_W+r_3+\mu_w)+(r_1+r_2)[\Omega N_W\delta\sigma(\delta+r_1+r_3+\mu_w)-(r_1+r_2+\mu_w)]+a^*}{\mu_w\delta^2\sigma^2(\delta+r_1+r_3+\mu_w)}\}$$

$S_W{}^*, O_0{}^*, O_1{}^*, R_W{}^*$ are positive when $\frac{\Omega N_W}{\delta+r_1+r_3+\mu_w} > \max\{k_1, k_2\}$
Where $k_1 = r_1 + r_2 + \mu_w$ and $k_2 = \gamma_W + r_3 + \mu_w$

5. Basic Reproduction Number

The basic reproduction number is calculated by the Next generation matrix method [6]
i.e., $R_0 = FV^{-1}$

$$\frac{\mathrm{dO_0}}{\mathrm{dt}} = \sigma S_W O_1 - (\delta + r_1 + r_3 + \mu_w) O_0$$
$$\frac{dO_1}{\mathrm{dt}} = \delta O_0 - (\gamma_W + r_3 + \mu_w) O_1 \qquad (6)$$

$$F = \begin{bmatrix} 0 & \sigma S_W \\ \delta O_0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -(\delta + r_1 + r_3 + \mu_w) & 0 \\ 0 & -(\gamma_W + r_3 + \mu_w) \end{bmatrix} \qquad (7)$$

At the disease free equilibrium

$$F = \begin{bmatrix} 0 & \dfrac{\sigma \Omega N_W}{\delta + r_1 + r_3 + \mu_w} \\ \delta & 0 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \dfrac{1}{(\gamma_W + r_3 + \mu_w)} & 0 \\ 0 & \dfrac{-1}{(\delta + r_1 + r_3 + \mu_w)} \end{bmatrix}$$

$$F\,V^{-1} = \begin{pmatrix} 0 & \dfrac{\sigma \Omega N_W}{(\delta + r_1 + r_3 + \mu_w)(\gamma_W + r_3 + \mu_w)} \\ \dfrac{-\sigma}{(\gamma_W + r_3 + \mu_w)^2} & 0 \end{pmatrix}$$

Thus the reproduction number of Osteoporosis

$$R_0 = \sqrt{\frac{\sigma \Omega N_W}{(\delta + r_1 + r_3 + \mu_w)(\gamma_W + r_3 + \mu_w)}}$$

6. Stability: Local behaviour

The Jacobian of (1) is

$$\begin{pmatrix} -(\sigma O_1 + \mu_w) & 0 & 0 & 0 \\ \sigma S_W & -(\delta + r_1 + r_3 + \mu_w) & \sigma S_W & 0 \\ 0 & \delta & -(\gamma_W + r_3 + \mu_w) & 0 \\ 0 & 0 & \sigma & -(\mu + d) \\ r_1 + r_2 & r_1 + r_3 & \gamma_W + r_3 & 0 \end{pmatrix} \qquad (8)$$

At the interior equilibrium

$$\begin{pmatrix} \dfrac{\Omega N_W}{S_W} & 0 & 0 & 0 \\ \sigma S_W & \dfrac{\sigma O_1 S_W}{O_0} & \sigma S_W & 0 \\ 0 & \delta & \dfrac{\delta O_0}{O_1} & 0 \\ r_1 + r_2 & r_1 + r_3 & \gamma_W + r_3 & \dfrac{(r_1 + r_2) S_W + (r_1 + r_3) O_0 + (\gamma_W + r_3) O_1}{R_W} \end{pmatrix} \qquad (9)$$

The characteristic equation of (9) is given by

$$\begin{vmatrix} \dfrac{\Omega N_W}{S_W} - \lambda & 0 & 0 & 0 \\ \sigma S_W & \dfrac{\sigma O_1 S_W}{O_0} - \lambda & \sigma S_W & 0 \\ 0 & \delta & \dfrac{\delta O_0}{O_1} - \lambda & 0 \\ r_1 + r_2 & r_1 + r_3 & \gamma_W + r_3 & \dfrac{(r_1 + r_2) S_W + (r_1 + r_3) O_0 + (\gamma_W + r_3) O_1}{R_W} - \lambda \end{vmatrix} = 0$$

$$\begin{aligned}
&\lambda^4+\Bigg(\frac{\Omega N_W}{S_W}+\frac{\delta O_0}{O_1}+\frac{\sigma S_W O_1}{O_0}+\frac{\delta O_0}{O_1}+\frac{(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1}{R_W}\Bigg)\lambda^3\\
&\quad+\Bigg(\frac{\Omega N_W\delta O_0}{S_W O_1}+\frac{\Omega N_W\sigma}{S_W O_0}+\frac{\sigma^2 O_1 {S_W}^2}{O_0}\\
&\quad+\frac{(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1}{R_W}\Bigg)\lambda^2\\
&\quad+\Bigg(\frac{\Omega N_W\sigma^2 O_1 S_W}{O_0}+\frac{\Omega N_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{O_1 R_W}\\
&\quad+\frac{\sigma\delta S_W[(r_1+r_2)S_W+(r_1+r_3)O_0]}{R_W}+\frac{\sigma\delta S_W(\gamma_W+r_3)O_1}{R_W}\\
&\quad+\frac{\Omega N_W\,\delta[(r_1+r_2)S_W+(r_1+r_3)O_0]}{S_W R_W}+\frac{\Omega N_W\,\delta(\gamma_W+r_3)O_0}{S_W R_W}\\
&\quad+\frac{\Omega N_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{S_W O_0 R_W}\\
&\quad+\frac{\sigma\delta S_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{R_W}\Bigg)\lambda\\
&\quad+\frac{\Omega N_W\sigma\delta[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{R_W}+\frac{\Omega N_W\,\sigma\delta O_1}{R_W}\\
&\quad+\frac{\Omega N_W\sigma^2\ [(r_1+r_2)S_W+(r_1+r_3)O_0}{O_0 R_W}+\Omega N_W\sigma\delta=0 \qquad \}10
\end{aligned}$$

Comparing (10) with $S^4+ES^3+FS^2+GS+H=0$

Where

$E=\frac{\Omega N_W}{S_W}+\frac{\delta O_0}{O_1}+\frac{\sigma S_W O_1}{O_0}+\frac{\delta O_0}{O_1}+\frac{(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1}{R_W}$

$F=\frac{\Omega N_W\delta O_0}{S_W O_1}+\frac{\Omega N_W\sigma}{S_W O_0}+\frac{\sigma^2 O_1 {S_W}^2}{O_0}+\frac{(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1}{R_W}$

$G=\frac{\Omega N_W\sigma^2 O_1 S_W}{O_0}+\frac{\Omega N_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{O_1 R_W}+\frac{\sigma\delta S_W[(r_1+r_2)S_W+(r_1+r_3)O_0]}{R_W}+\frac{\sigma\delta S_W(\gamma_W+r_3)O_1}{R_W}+$ $\frac{\Omega N_W\,\delta[(r_1+r_2)S_W+(r_1+r_3)O_0]}{S_W R_W}+\frac{\Omega N_W\,\delta(\gamma_W+r_3)O_0}{S_W R_W}+\frac{\Omega N_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{S_W O_0 R_W}+$ $\frac{\sigma\delta S_W[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{R_W}$

$H=\frac{\Omega N_W\sigma\delta[(r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)O_1]}{R_W}+\frac{\Omega N_W\,\sigma\delta O_1}{R_W}+\frac{\Omega N_W\sigma^2\ [(r_1+r_2)S_W+(r_1+r_3)O_0}{O_0 R_W}+\Omega N_W\sigma\delta$

$E>0$; $H>0$; $EF-G>0$; $G(EF-G)-HE^2>0$ when $1-S_W>0$.

Here, the model is stable locally by R.H. criteria.

7. Stability: Global behavior

Let

$V(S_W,\ O_0,\ O_1,\ R_W)=\ ((S_W-\ {S_W}^*)-\ \ {S_W}^*\ln\frac{S_W}{{S_W}^*}))\ +\ l_1((O_0-{O_0}^*)-\ {O_0}^*\ln\frac{O_0}{{O_0}^*}))\ +\ l_2((O_1-{O_1}^*)-$ ${O_1}^*\ln\frac{O_1}{{O_1}^*}))+l_3\ ((R_W-\ {R_W}^*)-\ {R_W}^*\ln\frac{R_W}{{R_W}^*}))$ (11)

Differentiate (11) with respect to t,

$\frac{dV}{dt}=\left(\frac{S_W-{S_W}^*}{S_W}\right)\frac{dS_W}{dt}+\left(\frac{O_0-\ {O_0}^*}{O_0}\right)\frac{dO_0}{dt}+\left(\frac{O_1-{O_1}^*}{O_1}\right)\frac{dO_1}{dt}+\left(\frac{R_W-\ {R_W}^*}{R_W}\right)\frac{dR_W}{dt}$

Using (1),

$\frac{dV}{dt}=\left(\frac{S_W-{S_W}^*}{S_W}\right)(\Omega N_W-(\sigma O_1+r_1+r_2+\mu_w)S_W)+\left(\frac{O_0-\ {O_0}^*}{O_0}\right)(\sigma S_W O_1-(\delta+r_1+r_3+\mu_w)O_0)+$ $\left(\frac{O_1-{O_1}^*}{O_1}\right)(\delta O_0-(\gamma_W+r_3+\mu_w)O_1)+\left(\frac{R_W-\ {R_W}^*}{R_W}\right)\left((r_1+r_2)S_W+(r_1+r_3)O_0+(\gamma_W+r_3)-\mu_w R_W\right)$

$$\frac{\mathrm{dV}}{\mathrm{dt}} = (\mathrm{S_W} - \mathrm{S_W}^*)\left(\frac{\Omega \mathrm{N_W}}{S_\mathrm{W}} - (\sigma O_1 + r_1 + r_2 + \mu_\mathrm{w})\right) + \mathrm{l}_1(\mathrm{O}_0 - \mathrm{O}_0{}^*)\left(\frac{\sigma \mathrm{S_W O_1}}{\mathrm{O}_0} - (\delta + r_1 + r_3 + \mu_\mathrm{w})\right)$$
$$+ \mathrm{l}_2(\mathrm{O}_1 - \mathrm{O}_1{}^*)\left(\frac{\delta \mathrm{O}_0}{\mathrm{O}_1} - (\gamma_\mathrm{W} + r_3 + \mu_\mathrm{w})\right)$$
$$+ \mathrm{l}_3(\mathrm{R_W} - \mathrm{R_W}^*)\left(\frac{(r_1 + r_2)S_\mathrm{W} + (r_1 + r_3)\mathrm{O}_0 + (\gamma_\mathrm{W} + r_3)}{\mathrm{R_W}} - \mu_\mathrm{w}\right)$$

At $(\mathrm{S_W}^*, \mathrm{O}_0{}^*, \mathrm{O}_1{}^*, \mathrm{R_W}^*)$ we have,

$$\frac{\mathrm{dV}}{\mathrm{dt}} = (\mathrm{S_W} - \mathrm{S_W}^*)\left[\frac{\Omega \mathrm{N_W}}{S_\mathrm{W}} - \left(\frac{\Omega \mathrm{N_W}}{\mathrm{S_W}^*}\right)\right] + \mathrm{l}_1(\mathrm{O}_0 - \mathrm{O}_0{}^*)\left(\frac{\sigma \mathrm{S_W O_1}}{\mathrm{O}_0} - \left(\frac{\sigma \mathrm{S_W O_1}^*}{\mathrm{O}_0{}^*}\right)\right) +$$
$$\mathrm{l}_2(\mathrm{O}_1 - \mathrm{O}_1{}^*)\left(\frac{\delta \mathrm{O}_0}{\mathrm{O}_1} - \frac{\delta \mathrm{O}_0{}^*}{\mathrm{O}_1}\right)$$
$$+ \mathrm{l}_3(\mathrm{R_W} - \mathrm{R_W}^*)\left(\frac{(r_1 + r_2)S_\mathrm{W} + (r_1 + r_3)\mathrm{O}_0 + (\gamma_\mathrm{W} + r_3)\mathrm{O}_1}{\mathrm{R_W}} - \frac{(r_1 + r_2)\mathrm{S_W}^* + (r_1 + r_3)\mathrm{O}_0{}^* + (\gamma_\mathrm{W} + r_3)\mathrm{O}_1{}^*}{\mathrm{R_W}^*}\right)$$

$$\frac{\mathrm{dV}}{\mathrm{dt}} = (\mathrm{S_W} - \mathrm{S_W}^*)\Omega \mathrm{N_W}\left[\frac{1}{S_\mathrm{W}} - \left(\frac{1}{\mathrm{S_W}^*}\right)\right] + \mathrm{l}_1(\mathrm{O}_0 - \mathrm{O}_0{}^*)\sigma\left(\frac{\mathrm{S_W O_1}}{\mathrm{O}_0} - \left(\frac{\mathrm{S_W}^* \mathrm{O}_1{}^*}{\mathrm{O}_0{}^*}\right)\right) +$$
$$\mathrm{l}_2(\mathrm{O}_1 - \mathrm{O}_1{}^*)\delta\left(\frac{\mathrm{O}_0}{\mathrm{O}_1} - \frac{\mathrm{O}_0{}^*}{\mathrm{O}_1{}^*}\right) + \mathrm{l}_3(\mathrm{R_W} - \mathrm{R_W}^*)(r_1 + r_2)\left(\frac{\mathrm{S_W}}{\mathrm{R_W}} - \frac{\mathrm{S_W}^*}{\mathrm{R_W}^*}\right)$$
$$+ \mathrm{l}_3(\mathrm{R_W} - \mathrm{R_W}^*)(r_1 + r_3)\left(\frac{\mathrm{O}_0}{\mathrm{R_W}} - \frac{\mathrm{O}_0{}^*}{\mathrm{R_W}^*}\right) + \mathrm{l}_3(\mathrm{R_W} - \mathrm{R_W}^*)(\gamma_\mathrm{W} + r_3)\left(\frac{\mathrm{O}_1}{\mathrm{R_W}} - \frac{\mathrm{O}_1{}^*}{\mathrm{R_W}^*}\right)$$

Choosing $\mathrm{l}_1 = \frac{1}{\sigma}, \mathrm{l}_2 = \frac{1}{\delta}, \mathrm{l}_3 = \frac{1}{(r_1 + r_2)(r_1 + r_3)(\gamma_\mathrm{W} + r_3)}$

$$\frac{\mathrm{dV}}{\mathrm{dt}} = (\mathrm{S_W} - \mathrm{S_W}^*)\Omega \mathrm{N_W}\left[\frac{1}{S_\mathrm{W}} - \left(\frac{1}{\mathrm{S_W}^*}\right)\right]$$
$$+ (\mathrm{O}_0 - \mathrm{O}_0{}^*)\left(\frac{\mathrm{S_W O_1}}{\mathrm{O}_0} - \left(\frac{\mathrm{S_W}^* \mathrm{O}_1{}^*}{\mathrm{O}_0{}^*}\right)\right)\mathrm{l}_2(\mathrm{O}_1 - \mathrm{O}_1{}^*)\delta\left(\frac{\mathrm{O}_0}{\mathrm{O}_1} - \frac{\mathrm{O}_0{}^*}{\mathrm{O}_1{}^*}\right)$$
$$+ \frac{(\mathrm{R_W} - \mathrm{R_W}^*)}{(r_1 + r_3)(\gamma_\mathrm{W} + r_3)}\left(\frac{\mathrm{S_W}}{\mathrm{R_W}} - \frac{\mathrm{S_W}^*}{\mathrm{R_W}^*}\right) + \frac{(\mathrm{R_W} - \mathrm{R_W}^*)}{(r_1 + r_2)(\gamma_\mathrm{W} + r_3)}\left(\frac{\mathrm{O}_0}{\mathrm{R_W}} - \frac{\mathrm{O}_0{}^*}{\mathrm{R_W}^*}\right)$$
$$+ \frac{(\mathrm{R_W} - \mathrm{R_W}^*)}{(r_1 + r_3)(r_1 + r_2)}\left(\frac{\mathrm{O}_1}{\mathrm{R_W}} - \frac{\mathrm{O}_1{}^*}{\mathrm{R_W}^*}\right)$$

$$\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{-\Omega \mathrm{N_W}(\mathrm{S_W} - \mathrm{S_W}^*)^2}{\mathrm{S_W S_W}^*} + \left(S_\mathrm{W}\mathrm{O}_0 - \frac{S_\mathrm{W}{}^*\mathrm{O}_0}{\mathrm{O}_0{}^*} - \frac{S_\mathrm{W}\mathrm{O}_0{}^*}{\mathrm{O}_0} + S_\mathrm{W}{}^*\mathrm{O}_0{}^*\right)$$
$$+ \left(\mathrm{O}_0 - \frac{\mathrm{O}_1{}^*\mathrm{O}_0}{\mathrm{O}_0{}^*} - \frac{\mathrm{O}_1\mathrm{O}_0{}^*}{\mathrm{O}_0} + \mathrm{O}_0{}^*\right)$$
$$+ \frac{1}{(r_1 + r_3)(\gamma_\mathrm{W} + r_3)}\left(S_\mathrm{W} - \frac{S_\mathrm{W}{}^*\mathrm{R_W}}{\mathrm{R_W}^*} - \frac{S_\mathrm{W}\mathrm{R_W}^*}{\mathrm{R_W}} + S_\mathrm{W}{}^*\right)$$
$$+ \frac{1}{(r_1 + r_2)(\gamma_\mathrm{W} + r_3)}\left(\mathrm{O}_0 - \frac{\mathrm{O}_0{}^*\mathrm{R_W}}{\mathrm{R_W}^*} - \frac{\mathrm{O}_0\mathrm{R_W}^*}{\mathrm{R_W}} + \mathrm{O}_0{}^*\right)$$
$$+ \frac{1}{(r_1 + r_3)(r_1 + r_2)}\left(\mathrm{O}_1 - \frac{\mathrm{O}_1{}^*\mathrm{R_W}}{\mathrm{R_W}^*} - \frac{\mathrm{O}_1\mathrm{R_W}^*}{\mathrm{R_W}} + \mathrm{O}_1{}^*\right)$$

$\therefore \frac{\mathrm{dV}}{\mathrm{dt}} < 0$, when $\frac{S_\mathrm{W}}{\mathrm{O}_0} < \frac{S_\mathrm{W}{}^*}{\mathrm{O}_0{}^*}, \frac{\mathrm{O}_0}{\mathrm{O}_1} < \frac{\mathrm{O}_0{}^*}{\mathrm{O}_1{}^*}, \frac{S_\mathrm{W}}{\mathrm{R_W}} < \frac{S_\mathrm{W}{}^*}{\mathrm{R_W}^*}, \frac{\mathrm{O}_0}{\mathrm{R_W}} < \frac{\mathrm{O}_0{}^*}{\mathrm{R_W}^*}, \frac{\mathrm{O}_1}{\mathrm{R_W}} < \frac{\mathrm{O}_1{}^*}{\mathrm{R_W}^*}$

Here, the model is stable globally by Lyapunov’s approach.

8. Sensitivity Analysis

The formula (12) to find the sensitive parameters is given below:

$$\xi_q^{R_0} = \frac{\partial R_0}{\partial a} \cdot \frac{a}{R_0}$$

Where ‘a’ indicates all the basic parameters.

Parameters	Sensitivity Index
Ω	+ve
μ_w	+ve
σ	+ve
r_1	-ve
r_2	-ve
r_3	-ve
γ_w	-ve
N_W	+ve

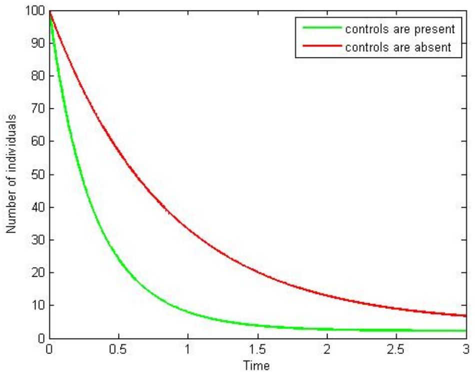
Table 1: Parameters and their Sensitivity indices

Table 1 shows the results of the sensitivity analysis for each parameter of the model. Ω, N_W, μ_w, δ and σ are positive parameters that have a high impact on osteoporosis.

r_1, r_2, r_3 and γ_w are the negative parameters that have a low impact on osteoporosis.

9. Numerical Simulations

We have chosen the parameter values Ω=0.09234, μ_W=0.042, σ=0.072, δ=0.020, γ_W=0.038.



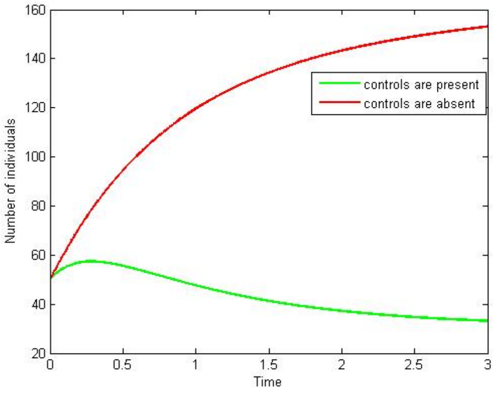


Figure 2: Susceptible with controls

Figure 3: Osteopenia with controls

Figure (2) represents the number of women in the susceptible class. The number of women in the susceptible class decreases when all control parameters exist. The number of women individuals in the susceptible class increases when all control parameters are absent.

Figure (3) shows the number of women in the osteopenia class. The number of women in the osteopenia class decreases when all control parameters are present. The number of women in the osteopenia class increases when all control parameters are absent.

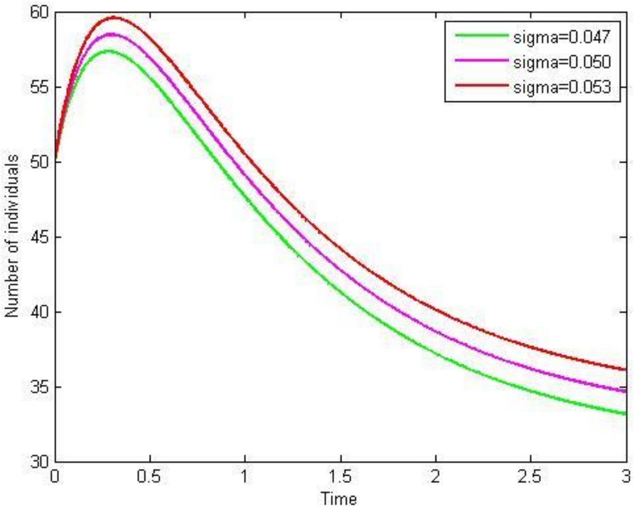


Figure 4: Osteopenia class for different valesof σ

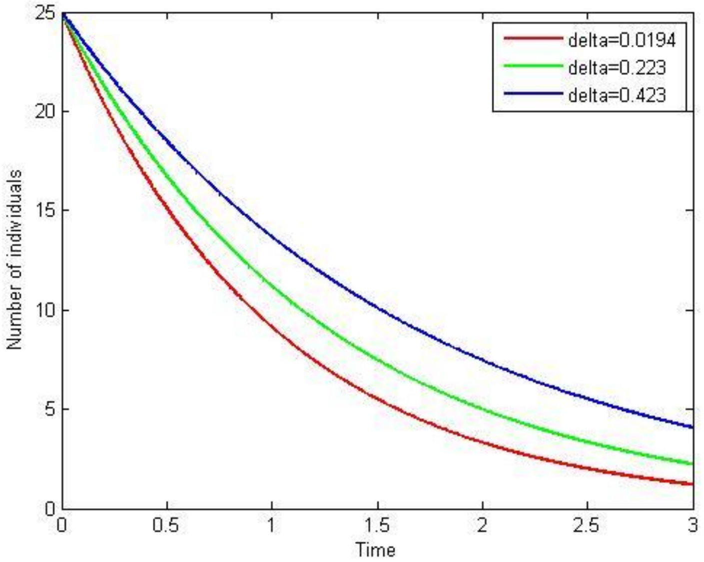


Figure 5: Osteopenia class for different valesof δ

Figure (4) depicts the number of women individuals of osteopenia class. When the value of σ increases, the number of women in the osteopenia class also decreases. Figure (5) shows the number of individuals in the osteoporosis class. When the δ value increases, the number of individuals in osteoporosis class also increases.

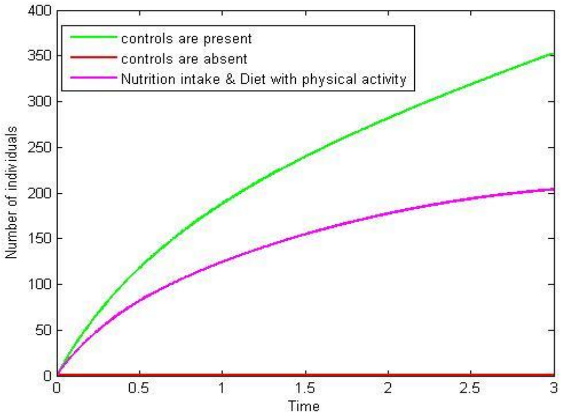


Figure 5: Recovered with controls

Figure (6) represents the number of women in the recovered class. When the control parameters are present, the number of women in the recovered class increases. Compared to the control parameter, diet with exercise and nutrition intake is involved, the number of women in the recovered class decreases. Recovery does not occur when all control parameters are absent.

8. Conclusion

In this paper, a mathematical model for osteoporosis is constructed with control parameters, namely a diet with exercise, nutrition intake and medication. The positivity and boundedness of the model are derived. The disease-free and endemic equilibrium points are determined. The reproduction number is found using the Next generation matrix. The system is found to be locally asymptotically stable around the endemic equilibrium through the Routh-Hurwitz criteria and globally asymptotically stable under a particular condition using the Lyapunov theorem. The sensitivity analysis is performed for all parameters involved in osteoporosis. Numerical simulations are carried out to demonstrate the behavior of different parameters such as the transition rate σ, recovery rate γ_w, exposed rate δ diet with exercise r_1, nutrition intake r_2, and medication r_3. The number of women individuals in the susceptible class decreases when all the control parameters exist. The number of women individuals in the susceptible class increases in the absence of all control parameters. The number of women individuals in the osteopenia class decreases when all control parameters are present. The number of women individuals in osteopenia class increases when all control parameters are absent. When the value of σ increases, the number of women individuals in the osteopenia class also decreases. When the control parameters are present, the number of women individuals of the recovered class increases. When control parameter diet with exercise and nutrition intake is involved the number of women individuals in recovered class decreases. Recovery does not occur when all control parameters are absent.

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