# An Overview of the Normalization and Cartesian Product on Intuitionistic Fuzzy Matrices

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***Abstract:*** In this paper, we present an Overview of the Normalization and Cartesian Product on Intuitionistic Fuzzy Matrices. In the case of discussion ordinary fuzzy matrices into normalization on intuitionistic fuzzy matrices. Also using the cartesian product x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11 we prove several important results for normalized intuitionistic fuzzy matrices.

Keywords: Fuzzy Matrices, Cartesian Product, Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Matrices, Normalization

## 1. Introduction

In 2014, A. Ejegwa, S.O. Akowe, P.M. Otene and J.M. Ikyule [1] introduced an Overview on Intuitionistic Fuzzy Sets. In 1965, Zadeh [2] introduced the concept of fuzzy sets as an extension to the theory of ordinary sets. In 1986, K. Atanassov [3] introduced the concept of intuitionistic fuzzy sets, fuzzy sets and systems. In 2002, M. Pal, S. K. Khan, A. K. Shymal [9] developed a theory for fuzzy matrices analogous to that for Intuitionistic fuzzy matrices, Intuitionistic Fuzzy Sets. So, the concept of intuitionistic fuzzy matrices is an extension of the concept of ordinary fuzzy matrices. In 1980, K. H. Kim and F. W. Roush [7] introduced the concept of Generalized fuzzy matrices to the theory of ordinary fuzzy sets by assigning to each element in the universe not only a membership degree but also a non-membership degree. In 2016, E. G. Emam, M. A. Fendh [8] introduced the concept of Some results associated with the max min and min max compositions of bi fuzzy matrices. In 2006 W. Zeng, H. Li [5] introduced the concept of some operations on intuitionistic fuzzy sets. In this paper, we define the normalization of intuitionistic fuzzy matrices and derive some results by using the cartesian product of intuitionistic fuzzy sets. However, we concentrate our studying to some kinds of intuitionistic fuzzy matrices, namely, normalization and normalization intuitionistic fuzzy matrices.
In this article, the intuitionistic fuzzy set approach using introduced to intuitionistic fuzzy matrices with normalized structures. Then a fuzzy set analytical algorithm is developed to aggregate decision attributes of alternatives for multicriteria decision making problems with intuitionistic fuzzy matrices and incomplete decision information. A series of non-linear programming models are constructed based on criteria weights intervals, belief degrees of fuzzy evidential reasoning analytical algorithm, then the genetic algorithm is employed to solve the non-linear models yielding the minimal and maximal fuzzy utilities of each alternative. With our anticipated method, procedures involving arithmetic operations is forementioned literature are not needed, thereby eradicating the limitations in those works.

## 2. Objectives:

We express the intuitionistic fuzzy matrices and some properties of the normalization intuitionistic fuzzy matrices are discussed. Intuitionistic fuzzy matrices play an important role in the field of fuzzy system modelling. Normalization Intuitionistic fuzzy matrices are extension of the intuitionistic fuzzy matrices. Eleven new operations are introduced over extended intuitionistic fuzzy matrices and over their simpler cases, such as max min or min max, extended normalization intuitionistic fuzzy matrices. Some properties of these ideas are discussed and statements are expressed.

Applications of Normalized Intuitionistic Fuzzy Matrices

Decision Making: When decision-makers face indecision and inconsistent criteria, Normalized Intuitionistic Fuzzy Matrices can be used to aggregate their preferences, which might be indefinite or fuzzy. For example, when choosing a supplier, each supplier's performance across multiple criteria (like cost, quality, and delivery time) is represented as an intuitionistic fuzzy set. The decision matrix is then normalized to make it reliable, and the best choice can be determined based on a multi-criteria decision-making method.

Multi-Criteria Decision Analysis: In Multi-Criteria Decision Analysis, decisions are taken by several criteria, each with indefinite or inaccurate data. Normalized Intuitionistic Fuzzy Matrices are used to represent these criteria by their corresponding membership and non-membership degrees. For example, when evaluating different products based on customer satisfaction, price, and quality, each product can be analysed using Normalized Intuitionistic Fuzzy Matrices for each criterion. Normalization ensures that the data is constant and facilitates assessments.

Pattern Recognition: In machine learning and pattern recognition, features of different classes or groups often contain uncertainty or overlap. Normalized Intuitionistic Fuzzy Matrices can be used to represent the features of each class with a degree of membership, non-membership, and indeterminacy. By normalizing the fuzzy matrices, the system can be more efficiently classified to patterns with undefined or imbrication characteristics.

Control Systems: In fuzzy control systems, the human operators or environments with uncertainty, Normalized Intuitionistic Fuzzy Matrices can help model and adjust control strategies. These matrices are used to represent control parameters where the control decision might have many possible outcomes with varying degrees of membership and non-membership. Normalization helps to guarantee that the fuzzy rules applied to the system do not lead to erratic or incompatible results.

Risk Analysis: In financial or project risk analysis, the level of risk for different situations can be expressed using intuitionistic fuzzy sets. Normalized Intuitionistic Fuzzy Matrices can then be used to represent and normalize these risks based on various limits like probability, impact, and uncertainty. This helps decision-makers to assess the potential risks and make informed choices.

Medical Diagnosis: In medical decision support systems, patient symptoms, diagnoses, and treatment methods can be represented using intuitionistic fuzzy sets. Normalized Intuitionistic Fuzzy Matrices help model the uncertainty essential in the medical data, where symptoms are not always perfectly matching a disease's profile. Normalized matrices are used to support diagnostic decisions based on uncertain medical data, such as in cases of erratic diseases or ambiguous symptoms.

Recommendation Systems: In recommendation systems, user preferences, item attributes, and recommendation scores are often uncertain or imprecise. Intuitionistic fuzzy sets can represent these uncertainties, and Normalized Intuitionistic Fuzzy Matrices are used to normalize the scores and generate personalized recommendations. This is particularly useful in contexts where user preferences are indefinite or evolving, such as in online shopping, entertainment, or social media.

Data Synthesis: In situations where data from multiple sources is combined, normalization helps to adjust different data sources into a uniform structure, making it easier to collect the information.

## 3. Preliminaries

Definition 3.1 A fuzzy set is any set that allows its members to have different degree of membership function, having interval [0, 1].

Definition 3.2 Fuzzy matrices play a vital role in scientific development. A Fuzzy matrix may be matrix that has its parts from [0, 1]. Consider a matrix $A = [aij]3\times3$ where $aij \in [0,1]$, $1 \leq j \leq n$. Then A is a Fuzzy Matrix.

Definition 3.3 An intuitionistic fuzzy set A in E is defined as an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$ where the functions $\mu_A : E \to [0,1]$ define the membership and the degree of non-membership of the element$x \in E$, respectively, and for every $x \in E$, $0 \leq \mu_A(x)\nu_A(x) \leq 1$. obviously, each ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$.

Definition 3.4 Intuitionistic Fuzzy Matrices: Let A= $[a_{ij}]_{m\times n}$ and A' = $[a'_{ij}]_{m\times n}$ be two fuzzy matrices such that $a_{ij}$+ $a'_{ij}$ $\leq$1for every i$\leq$ m and j$\leq$ n. The pair $\langle A, A'\rangle$ is called an intuitionistic fuzzy matrix and is denoted by B and then we write B = $\left[b_{ij} = \langle a_{ij}, a'_{ij}\rangle\right]_{m\times n}$.

Definition 3.5 Let x be a nonempty universal set. The normalization of an intuitionistic fuzzy set A denoted by NORM(A) is defined as NORM(A) = $\{\langle x, \mu_{NORM(A)}(x), \vartheta_{NORM(A)}(x)\rangle : x \in X\}$ where $\mu_{NORM(A)}(x) = \frac{\mu_A(x)}{\sup(\mu_A(x)}$ and $\vartheta_{NORM(A)}(x) = \frac{\vartheta_A(x)-\inf(\vartheta_A(x))}{1-\inf(\vartheta_A(x))}$ for X = {x}.

## 4. The Cartesian Products of Intuitionistic Fuzzy Sets

Definition 4.1 The Cartesian Products of two NIFSs A and B are defined as follows,
The Cartesian product "$\times_1$" then
$A \times_1 B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.2 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product "$\times_2$" then
$A \times_2 B = \{\langle\langle x, y\rangle, \mu_A(x) + \mu_B(y) - \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.3 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product "$\times_3$" then
$A \times_3 B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x) + \nu_B(y) - \nu_A(x).\nu_B(y)\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.4 The Cartesian products of two NIFSs A and B are defined as follows, The Cartesian product "$\times_4$" then
$A \times_4 B = \{\langle\langle x, y\rangle, \min(\mu_A(x), \mu_B(y)), \max(\nu_A(x), \nu_B(y))\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.5 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product "$\times_5$" then
$A \times_5 B = \{\langle\langle x, y\rangle, \max(\mu_A(x), \mu_B(y)), \min(\nu_A(x), \nu_B(y))\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.6 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_6$" then
$A \times_6 B = \{\langle\langle x, y\rangle, \left(\frac{\mu_A(x) + \mu_B(y)}{2}, \frac{\nu_A(x) + \nu_B(y)}{2}\right)\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.7 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_7$" then
$A \times_7 B = \{\langle\langle x, y\rangle, \min(1, \mu_A(x) + \mu_B(y)), \max(0, \nu_A(x) + \nu_B(y) - 1)\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.8 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_8$" then
$A \times_8 B = \{\langle\langle x, y\rangle, \max(0, \mu_A(x) + \mu_B(y) - 1), \min(1, \nu_A(x) + \nu_B(y))\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.9 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_9$" then
$A \times_9 B = \{\langle\langle x, y\rangle, \sqrt{\mu_A(x)\,\mu_B(y)}, \sqrt{\nu_A(x)\nu_B(y)}\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.10 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_{10}$" then
$A \times_{10} B = \{\langle\langle x, y\rangle, 2\frac{\mu_A(x).\ \mu_B(y)}{\mu_A(x) + \mu_B(y)}, 2\frac{\nu_A(x).\ \nu_B(y)}{\nu_A(x) + \nu_B(y)}\rangle \mid x \in E_1 \& y \in E_2\}$.
Definition 4.11 The Cartesian products of two NIFSs A and B are defined as follows,
The Cartesian product of "$\times_{11}$" then
$A \times_{11} B = \{\langle\langle x, y\rangle, \frac{\mu_A(x) + \mu_B(y)}{2(\mu_A(x)\,\mu_B(y)+1)}, \frac{\nu_A(x) + \nu_B(y)}{2(\nu_A(x)\nu_B(y)+1)}\rangle \mid x \in E_1 \& y \in E_2\}$.

## 5. Results:

Main theorem of Normalization on Intuitionistic Fuzzy Matrices Theorem 5.1: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_1 B$ is also normed intuitionistic fuzzy matrix.
Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

$$\text{If } A = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$$

$$B=\begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$$ is also are normed intuitionistic fuzzy matrix. Now applying the cartesian product of "$\times_1$" on both sides, we have

$$A\times_1 B=\begin{pmatrix}(\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x))\\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x))\end{pmatrix}\times_1 \begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_1 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_1 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_1 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_1 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_1 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_1 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_1 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_1 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_1 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_1 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_1 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_1 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of cartesian product of "$\times_1$" by

$$A\times_1 B=\{\langle\langle x, y\rangle, \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x)}\cdot\frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y)}, \frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))}\cdot\frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))}\rangle \mid x\in E_1 \ \&\ y\in E_2\}.$$

$$x_{11}=\left(\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))}\cdot\frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}, \frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))}\cdot\frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)$$

$$x_{12}=\left(\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))}\cdot\frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}, \frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))}\cdot\frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)$$

$$x_{21}=\left(\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))}\cdot\frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}, \frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))}\cdot\frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)$$

$$x_{22}=\left(\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))}\cdot\frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}, \frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))}\cdot\frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)$$

We have, $A \times_1 B = \begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.2. If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_2 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If $$A=\begin{pmatrix}(\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x))\\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x))\end{pmatrix}$$

$$B=\begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$$ is also are normed intuitionistic fuzzy matrix. Now applying the cartesian product of "$\times_2$" on both sides, we have

$$A\times_2 B=\begin{pmatrix}(\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x))\\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x))\end{pmatrix}\times_2 \begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_2 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_2 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_2 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_2 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_2 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_2 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_2 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_2 \vartheta_{NORM(\rho_{12})}(y))$

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$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_2 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_2 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of cartesian product of "$\times_2$" by

$$A\times_2 B=\{\langle\langle x, y\rangle \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x)} + \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))} - \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x)} \cdot \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}, \frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))} \cdot \frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))}\rangle \mid x \in E_1 \ \&\ y \in E_2\}.$$

$$x_{11} = \left(\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x)} + \frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))} - \frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x)} \cdot \frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}, \frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)$$

$$x_{12} = \left(\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x)} + \frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))} - \frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x)} \cdot \frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}, \frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)$$

$$x_{21} = \left(\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x)} + \frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))} - \frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x)} \cdot \frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}, \frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)$$

$$x_{22} = \left(\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x)} + \frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))} - \frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x)} \cdot \frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}, \frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)$$

We have $A \times_2 B = \begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5. 3: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_3 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If $A = \begin{pmatrix}(\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x))\\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x))\end{pmatrix}$

$B = \begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the cartesian product of "$\times_3$" on both sides, we have

$$A \times_3 B = \begin{pmatrix}(\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x))\\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x))\end{pmatrix}\times_3 \begin{pmatrix}(\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y))\\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y))\end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_3 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_3 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_3 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_3 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_3 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_3 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_3 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_3 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_3 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_3 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_3 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_3 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of cartesian product of "$\times_3$" by

$$A \times_3 B = \{\langle\langle x, y\rangle, \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))} \cdot \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y)},\ \frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))} + \frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))} - \frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))} \cdot \frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))}\rangle \mid x \in E_1 \ \&\ y \in E_2\}.$$

$$x_{11} = \left(\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))} \cdot \frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y)},\ \frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))} + \frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))} - \frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)$$

$$x_{12} = \left(\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))} \cdot \frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y)},\ \frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))} + \frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))} - \frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)$$

$$x_{21} = \left(\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))} \cdot \frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y)},\ \frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))} + \frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))} - \frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)$$

$$x_{22} = \left(\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))} \cdot \frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y)},\ \frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))} + \frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))} - \frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))} \cdot \frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)$$

We have $A \times_3 B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.4: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_4 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If $A = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

$B = \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the cartesian product of "$\times_4$" on both sides, we have

$$A \times_4 B = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_4 \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_4 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_4 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_4 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$
$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_4 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$
$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_4 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_4 \vartheta_{NORM(\rho_{11})}(y))$
$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_4 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_4 \vartheta_{NORM(\rho_{12})}(y))$
$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_4 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_4 \vartheta_{NORM(\rho_{21})}(y))$
$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_4 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_4 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of cartesian product of "$\times_4$" by

$$A\times_4 B=\{\langle\langle x,y\rangle,\min\left(\frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))},\frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}\right),\max\left(\frac{\vartheta_{NORM(A)}(x)-\inf\left(\vartheta_{NORM(A)}(x)\right)}{1-\inf\left(\vartheta_{NORM(A)}(x)\right)},\frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))}\right)\rangle \mid x \in E_1 \ \&\ y \in E_2\}.$$

$$x_{11}=\left(\min\left(\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))},\frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}\right),\max\left(\frac{\vartheta_{NORM(\theta_{11})}(x)-\inf\left(\vartheta_{NORM(\theta_{11})}(x)\right)}{1-\inf\left(\vartheta_{NORM(\theta_{11})}(x)\right)},\frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)\right)$$

$$x_{12}=\left(\min\left(\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))},\frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}\right),\max\left(\frac{\vartheta_{NORM(\theta_{12})}(x)-\inf\left(\vartheta_{NORM(\theta_{12})}(x)\right)}{1-\inf\left(\vartheta_{NORM(\theta_{12})}(x)\right)},\frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)\right)$$

$$x_{21}=\left(\min\left(\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))},\frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}\right),\max\left(\frac{\vartheta_{NORM(\theta_{21})}(x)-\inf\left(\vartheta_{NORM(\theta_{21})}(x)\right)}{1-\inf\left(\vartheta_{NORM(\theta_{21})}(x)\right)},\frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)\right)$$

$$x_{22}=\left(\min\left(\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))},\frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}\right),\max\left(\frac{\vartheta_{NORM(\theta_{22})}(x)-\inf\left(\vartheta_{NORM(\theta_{22})}(x)\right)}{1-\inf\left(\vartheta_{NORM(\theta_{22})}(x)\right)},\frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)\right)$$

We have $A \times_4 B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.5: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_5 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If $A = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

$B= \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_5$" on both sides, we have

$$A\times_5 B=\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_5 \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_5 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$
$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_5 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$
$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_5 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_5 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_5 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_5 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_5 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_5 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_5 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_5 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_5 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_5 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of Cartesian product of "$\times_5$" by

$$A\times_5 B=\{\langle\langle x,y\rangle, \max(\frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))},\frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}), \min(\frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))}, \frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))})\rangle \mid x \in E_1 \ \&\ y \in E_2\}.$$

$$x_{11}=\left(\max\left(\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))},\frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}\right), \min\left(\frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))},\frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)\right)$$

$$x_{12}=\left(\max\left(\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))},\frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}\right), \min\left(\frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))},\frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)\right)$$

$$x_{21}=\left(\max\left(\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))},\frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}\right), \min\left(\frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))},\frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)\right)$$

$$x_{22}=\left(\max\left(\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))},\frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}\right), \min\left(\frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))},\frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)\right)$$

We have $A \times_5 B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.6: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_6 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If $A = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

$B= \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_6$" on both sides, we have

$$A\times_6 B=\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_6 \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_6 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_6 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_6 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_6 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_6 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_6 \vartheta_{NORM(\rho_{11})}(y))$
$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_6 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_6 \vartheta_{NORM(\rho_{12})}(y))$
$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_6 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_6 \vartheta_{NORM(\rho_{21})}(y))$
$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_6 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_6 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of Cartesian product of "$\times_6$" by

$$A \times_6 B = \{\langle\langle x, y\rangle, \left(\frac{\frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))} + \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}}{2}\right), \left(\frac{\frac{\vartheta_{NORM(A)}(x) - \inf(\vartheta_{NORM(A)}(x))}{1 - \inf(\vartheta_{NORM(A)}(x))} + \frac{\vartheta_{NORM(B)}(y) - \inf(\vartheta_{NORM(B)}(y))}{1 - \inf(\vartheta_{NORM(B)}(y))}}{2}\right)\rangle \mid x \in E_1 \;\&\; y \in E_2\}.$$

$$x_{11} = \left(\left(\frac{\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))} + \frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}}{2}\right), \left(\frac{\frac{\vartheta_{NORM(\theta_{11})}(x) - \inf(\vartheta_{NORM(\theta_{11})}(x))}{1 - \inf(\vartheta_{NORM(\theta_{11})}(x))} + \frac{\vartheta_{NORM(\rho_{11})}(y) - \inf(\vartheta_{NORM(\rho_{11})}(y))}{1 - \inf(\vartheta_{NORM(\rho_{11})}(y))}}{2}\right)\right)$$

$$x_{12} = \left(\left(\frac{\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))} + \frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}}{2}\right), \left(\frac{\frac{\vartheta_{NORM(\theta_{12})}(x) - \inf(\vartheta_{NORM(\theta_{12})}(x))}{1 - \inf(\vartheta_{NORM(\theta_{12})}(x))} + \frac{\vartheta_{NORM(\rho_{12})}(y) - \inf(\vartheta_{NORM(\rho_{12})}(y))}{1 - \inf(\vartheta_{NORM(\rho_{12})}(y))}}{2}\right)\right)$$

$$x_{21} = \left(\left(\frac{\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))} + \frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}}{2}\right), \left(\frac{\frac{\vartheta_{NORM(\theta_{21})}(x) - \inf(\vartheta_{NORM(\theta_{21})}(x))}{1 - \inf(\vartheta_{NORM(\theta_{21})}(x))} + \frac{\vartheta_{NORM(\rho_{21})}(y) - \inf(\vartheta_{NORM(\rho_{21})}(y))}{1 - \inf(\vartheta_{NORM(\rho_{21})}(y))}}{2}\right)\right)$$

$$x_{22} = \left(\left(\frac{\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))} + \frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}}{2}\right), \left(\frac{\frac{\vartheta_{NORM(\theta_{22})}(x) - \inf(\vartheta_{NORM(\theta_{22})}(x))}{1 - \inf(\vartheta_{NORM(\theta_{22})}(x))} + \frac{\vartheta_{NORM(\rho_{22})}(y) - \inf(\vartheta_{NORM(\rho_{22})}(y))}{1 - \inf(\vartheta_{NORM(\rho_{22})}(y))}}{2}\right)\right)$$

We have $A \times_6 B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.7: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then $A \times_7 B$ is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If A = $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

B= $\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_7$" on both sides, we have

A$\times_7$B= $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_7$

$\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_7 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_7 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_7 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_7 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_7 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_7 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_7 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_7 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_7 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_7 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_7 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_7 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of Cartesian product of "$\times_7$" by

$A \times_7 B = \{\langle\langle x, y\rangle, \min(1, \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))} + \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}),\ \max(0, \frac{\vartheta_{NORM(A)}(x) - \inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))} + \frac{\vartheta_{NORM(B)}(y) - \inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))} - 1)\rangle \mid x \in E_1\ \&\ y \in E_2\}.$

$$x_{11} = \left(\min\left(1, \frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup\left(\mu_{NORM(\epsilon_{11})}(x)\right)} + \frac{\mu_{NORM(\psi_{11})}(y)}{\sup\left(\mu_{NORM(\psi_{11})}(y)\right)}\right),\right.$$
$$\left.\left(\max\left(0, \frac{\vartheta_{NORM(\theta_{11})}(x) - \inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))} + \frac{\vartheta_{NORM(\rho_{11})}(y) - \inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))} - 1\right)\right)\right)$$

$$x_{12} = \left(\min\left(1, \frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup\left(\mu_{NORM(\epsilon_{12})}(x)\right)} + \frac{\mu_{NORM(\psi_{12})}(y)}{\sup\left(\mu_{NORM(\psi_{12})}(y)\right)}\right),\right.$$
$$\left.\left(\max\left(0, \frac{\vartheta_{NORM(\theta_{12})}(x) - \inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))} + \frac{\vartheta_{NORM(\rho_{12})}(y) - \inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))} - 1\right)\right)\right)$$

$$x_{21} = \left(\min\left(1, \frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup\left(\mu_{NORM(\epsilon_{21})}(x)\right)} + \frac{\mu_{NORM(\psi_{21})}(y)}{\sup\left(\mu_{NORM(\psi_{21})}(y)\right)}\right),\right.$$
$$\left.\left(\max\left(0, \frac{\vartheta_{NORM(\theta_{21})}(x) - \inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))} + \frac{\vartheta_{NORM(\rho_{21})}(y) - \inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))} - 1\right)\right)\right)$$

$$x_{22} = \left(\min\left(1, \frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup\left(\mu_{NORM(\epsilon_{22})}(x)\right)} + \frac{\mu_{NORM(\psi_{22})}(y)}{\sup\left(\mu_{NORM(\psi_{22})}(y)\right)}\right),\right.$$
$$\left.\left(\max\left(0, \frac{\vartheta_{NORM(\theta_{22})}(x) - \inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))} + \frac{\vartheta_{NORM(\rho_{22})}(y) - \inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))} - 1\right)\right)\right)$$

We have $A \times_7 B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.8: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then A $\times_8$ B is also normed intuitionistic fuzzy matrix.
Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If A = $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

B= $\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_8$" on both sides, we have

$$A\times_8 B= \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_8 \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_8 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_8 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_8 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_8 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_8 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_8 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_8 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_8 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_8 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_8 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_8 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_8 \vartheta_{NORM(\rho_{22})}(y))$

By us ing formula of Cartesian product of "$\times_8$" by

$A \times_8 B = \{\langle\langle x, y\rangle, \max(0, \frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))} + \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))} - 1), \min(1, \frac{\vartheta_{NORM(A)}(x) - \inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))} + \frac{\vartheta_{NORM(B)}(y) - \inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))})\rangle \mid x \in E_1 \ \&\ y \in E_2\}$.

$$x_{11} = \left(\max\left(0, \frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup\left(\mu_{NORM(\epsilon_{11})}(x)\right)} + \frac{\mu_{NORM(\psi_{11})}(y)}{\sup\left(\mu_{NORM(\psi_{11})}(y)\right)} - 1\right), \left(\min\left(1, \frac{\vartheta_{NORM(\theta_{11})}(x) - \inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))} + \frac{\vartheta_{NORM(\rho_{11})}(y) - \inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}\right)\right)\right)$$

$$x_{12} = \left(\max\left(0, \frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup\left(\mu_{NORM(\epsilon_{12})}(x)\right)} + \frac{\mu_{NORM(\psi_{12})}(y)}{\sup\left(\mu_{NORM(\psi_{12})}(y)\right)} - 1\right), \left(\min\left(1, \frac{\vartheta_{NORM(\theta_{12})}(x) - \inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))} + \frac{\vartheta_{NORM(\rho_{12})}(y) - \inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}\right)\right)\right)$$

$$x_{21} = \left(\max\left(0, \frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup\left(\mu_{NORM(\epsilon_{21})}(x)\right)} + \frac{\mu_{NORM(\psi_{21})}(y)}{\sup\left(\mu_{NORM(\psi_{21})}(y)\right)} - 1\right), \left(\min\left(1, \frac{\vartheta_{NORM(\theta_{21})}(x) - \inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))} + \frac{\vartheta_{NORM(\rho_{21})}(y) - \inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}\right)\right)\right)$$

$$x_{22} = \left(\max\left(0, \frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup\left(\mu_{NORM(\epsilon_{22})}(x)\right)} + \frac{\mu_{NORM(\psi_{22})}(y)}{\sup\left(\mu_{NORM(\psi_{22})}(y)\right)} - 1\right),\right.$$

$$\left(\min\left(1,\frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))}+\frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}\right)\right)\right)$$

We have A $\times_8$ B = $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.9: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then A $\times_9$ B is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If A = $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

B= $\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_9$" on both sides, we have

A$\times_9$B= $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_9$
$\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_9 (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_9 (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_9 (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_9 (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_9 \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_9 \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_9 \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_9 \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_9 \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_9 \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_9 \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_9 \vartheta_{NORM(\rho_{22})}(y))$

By using formula of Cartesian product of "$\times_9$" by A $\times_9$ B = $\{\langle\langle x, y\rangle, \sqrt{\frac{\mu_{NORM(A)}(x)}{\sup(\mu_{NORM(A)}(x))}\ \frac{\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(B)}(y))}}, \sqrt{\frac{\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))}{1-\inf(\vartheta_{NORM(A)}(x))}\ \frac{\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y))}{1-\inf(\vartheta_{NORM(B)}(y))}}\rangle \mid x \in E_1\ \&\ y \in E_2\}$.

$$\mathbf{x_{11}} = \left(\sqrt{\frac{\mu_{NORM(\epsilon_{11})}(x)}{\sup(\mu_{NORM(\epsilon_{11})}(x))}\ \frac{\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\psi_{11})}(y))}},\ \sqrt{\frac{\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))}{1-\inf(\vartheta_{NORM(\theta_{11})}(x))}\ \frac{\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y))}{1-\inf(\vartheta_{NORM(\rho_{11})}(y))}}\right)$$

$$\mathbf{x_{12}} = \left(\sqrt{\frac{\mu_{NORM(\epsilon_{12})}(x)}{\sup(\mu_{NORM(\epsilon_{12})}(x))}\ \frac{\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\psi_{12})}(y))}},\ \sqrt{\frac{\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))}{1-\inf(\vartheta_{NORM(\theta_{12})}(x))}\ \frac{\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y))}{1-\inf(\vartheta_{NORM(\rho_{12})}(y))}}\right)$$

$$\mathbf{x_{21}} = \left(\sqrt{\frac{\mu_{NORM(\epsilon_{21})}(x)}{\sup(\mu_{NORM(\epsilon_{21})}(x))}\ \frac{\mu_{NORM(\psi_{21})}(y)}{\sup(\mu_{NORM(\psi_{21})}(y))}},\right.$$

$$\sqrt{\frac{\vartheta_{NORM(\theta_{21})}(x)-\inf(\vartheta_{NORM(\theta_{21})}(x))}{1-\inf(\vartheta_{NORM(\theta_{21})}(x))}\frac{\vartheta_{NORM(\rho_{21})}(y)-\inf(\vartheta_{NORM(\rho_{21})}(y))}{1-\inf(\vartheta_{NORM(\rho_{21})}(y))}}\Bigg)$$

$$x_{22}=\Bigg(\sqrt{\frac{\mu_{NORM(\epsilon_{22})}(x)}{\sup(\mu_{NORM(\epsilon_{22})}(x))}\frac{\mu_{NORM(\psi_{22})}(y)}{\sup(\mu_{NORM(\psi_{22})}(y))}},$$

$$\sqrt{\frac{\vartheta_{NORM(\theta_{22})}(x)-\inf(\vartheta_{NORM(\theta_{22})}(x))}{1-\inf(\vartheta_{NORM(\theta_{22})}(x))}\frac{\vartheta_{NORM(\rho_{22})}(y)-\inf(\vartheta_{NORM(\rho_{22})}(y))}{1-\inf(\vartheta_{NORM(\rho_{22})}(y))}}\Bigg)$$

We have A $\times_9$ B = $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5. 10: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then A $\times_{10}$ B is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If A = $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

B= $\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_{10}$" on both sides, we have

$$A\times_{10}B=\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}\times_{10}\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ , where

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_{10} (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_{10} (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_{10} (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_{10} (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$

$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_{10} \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_{10} \vartheta_{NORM(\rho_{11})}(y))$

$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_{10} \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_{10} \vartheta_{NORM(\rho_{12})}(y))$

$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_{10} \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_{10} \vartheta_{NORM(\rho_{21})}(y))$

$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_{10} \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_{10} \vartheta_{NORM(\rho_{22})}(y))$

By using formula of Cartesian product of "$\times_{10}$" by

$$A \times_{10} B = \{\langle\langle x, y\rangle, 2\frac{\mu_{NORM(A)}(x)\cdot\mu_{NORM(B)}(y)}{\sup(\mu_{NORM(A)}(x))+\sup(\mu_{NORM(B)}(y))}, 2\frac{(\vartheta_{NORM(A)}(x)-\inf(\vartheta_{NORM(A)}(x))).(\vartheta_{NORM(B)}(y)-\inf(\vartheta_{NORM(B)}(y)))}{(1-\inf(\vartheta_{NORM(A)}(x)))+(1-\inf(\vartheta_{NORM(B)}(y)))}\rangle \mid x\in E_1 \ \&\ y\in E_2\}.$$

$$x_{11}=\left(2\frac{\mu_{NORM(\epsilon_{11})}(x)\cdot\mu_{NORM(\psi_{11})}(y)}{\sup(\mu_{NORM(\epsilon_{11})}(x))+\sup(\mu_{NORM(\psi_{11})}(y))},\ 2\frac{(\vartheta_{NORM(\theta_{11})}(x)-\inf(\vartheta_{NORM(\theta_{11})}(x))).(\vartheta_{NORM(\rho_{11})}(y)-\inf(\vartheta_{NORM(\rho_{11})}(y)))}{(1-\inf(\vartheta_{NORM(\theta_{11})}(x)))+(1-\inf(\vartheta_{NORM(\rho_{11})}(y)))}\right)$$

$$x_{12}=\left(2\frac{\mu_{NORM(\epsilon_{12})}(x)\cdot\mu_{NORM(\psi_{12})}(y)}{\sup(\mu_{NORM(\epsilon_{12})}(x))+\sup(\mu_{NORM(\psi_{12})}(y))},\ 2\frac{(\vartheta_{NORM(\theta_{12})}(x)-\inf(\vartheta_{NORM(\theta_{12})}(x))).(\vartheta_{NORM(\rho_{12})}(y)-\inf(\vartheta_{NORM(\rho_{12})}(y)))}{(1-\inf(\vartheta_{NORM(\theta_{12})}(x)))+(1-\inf(\vartheta_{NORM(\rho_{12})}(y)))}\right)$$

$$\boldsymbol{x_{21}} = \left( \mathbf{2}\frac{\mu_{NORM(\epsilon_{21})}(x) \cdot \mu_{NORM(\psi_{21})}(y)}{sup(\mu_{NORM(\epsilon_{21})}(x)) + sup\,(\mu_{NORM(\psi_{21})}(y))}, \mathbf{2}\frac{\left(\vartheta_{NORM(\theta_{21})}(x) - inf\,(\vartheta_{NORM(\theta_{21})}(x))\right).\left(\vartheta_{NORM(\rho_{21})}(y) - inf\,(\vartheta_{NORM(\rho_{21})}(y))\right)}{\left(1 - inf\left(\vartheta_{NORM(\theta_{21})}(x)\right)\right) + \left(1 - inf\left(\vartheta_{NORM(\rho_{21})}(y)\right)\right)} \right)$$

$$\boldsymbol{x_{22}} = \left( \mathbf{2}\frac{\mu_{NORM(\epsilon_{22})}(x) \cdot \mu_{NORM(\psi_{22})}(y)}{sup(\mu_{NORM(\epsilon_{22})}(x)) + sup\,(\mu_{NORM(\psi_{22})}(y))}, \mathbf{2}\frac{\left(\vartheta_{NORM(\theta_{22})}(x) - inf\,(\vartheta_{NORM(\theta_{22})}(x))\right).\left(\vartheta_{NORM(\rho_{22})}(y) - inf\,(\vartheta_{NORM(\rho_{22})}(y))\right)}{\left(1 - inf\left(\vartheta_{NORM(\theta_{22})}(x)\right)\right) + \left(1 - inf\left(\vartheta_{NORM(\rho_{22})}(y)\right)\right)} \right)$$

We have A $\times_{10}$ B = $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

Theorem 5.11: If E and F be two universal sets. For every normalization of an intuitionistic fuzzy matrices are in E and F then A $\times_{\mathbf{11}}$ B is also normed intuitionistic fuzzy matrix.

Proof: Let us consider A and B are two 2×2 normalization of an intuitionistic fuzzy matrices.

If A = $\begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix}$

B= $\begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$ is also are normed intuitionistic fuzzy matrix. Now applying the Cartesian product of "$\times_{\mathbf{11}}$" on both sides, we have

$$A\times_{11}B = \begin{pmatrix} (\mu_{NORM(\epsilon_{11})}(x) & \vartheta_{NORM(\theta_{11})}(x)) & (\mu_{NORM(\epsilon_{12})}(x) & \vartheta_{NORM(\theta_{12})}(x)) \\ (\mu_{NORM(\epsilon_{21})}(x) & \vartheta_{NORM(\theta_{21})}(x)) & (\mu_{NORM(\epsilon_{22})}(x) & \vartheta_{NORM(\theta_{22})}(x)) \end{pmatrix} \times_{\mathbf{11}} \begin{pmatrix} (\mu_{NORM(\psi_{11})}(y) & \vartheta_{NORM(\rho_{11})}(y)) & (\mu_{NORM(\psi_{12})}(y) & \vartheta_{NORM(\rho_{12})}(y)) \\ (\mu_{NORM(\psi_{21})}(y) & \vartheta_{NORM(\rho_{21})}(y)) & (\mu_{NORM(\psi_{22})}(y) & \vartheta_{NORM(\rho_{22})}(y)) \end{pmatrix}$$

Calculate, $\mathbf{x_{11}}$, $\mathbf{x_{12}}$, $\mathbf{x_{21}}$ and $\mathbf{x_{22}}$ , where

$$\mathbf{x_{11}} = (\mu_{NORM(\epsilon_{11})}(x), \vartheta_{NORM(\theta_{11})}(x)) \times_{11} (\mu_{NORM(\psi_{11})}(y), \vartheta_{NORM(\rho_{11})}(y))$$

$$x_{12} = (\mu_{NORM(\epsilon_{12})}(x), \vartheta_{NORM(\theta_{12})}(x)) \times_{11} (\mu_{NORM(\psi_{12})}(y), \vartheta_{NORM(\rho_{12})}(y))$$

$$x_{21} = (\mu_{NORM(\epsilon_{21})}(x), \vartheta_{NORM(\theta_{21})}(x)) \times_{11} (\mu_{NORM(\psi_{21})}(y), \vartheta_{NORM(\rho_{21})}(y))$$

$$x_{22} = (\mu_{NORM(\epsilon_{22})}(x), \vartheta_{NORM(\theta_{22})}(x)) \times_{11} (\mu_{NORM(\psi_{22})}(y), \vartheta_{NORM(\rho_{22})}(y))$$

$$x_{11} = (\mu_{NORM(\epsilon_{11})}(x) \times_{11} \mu_{NORM(\psi_{11})}(y)),\ (\vartheta_{NORM(\theta_{11})}(x)) \times_{11} \vartheta_{NORM(\rho_{11})}(y))$$

$$x_{12} = (\mu_{NORM(\epsilon_{12})}(x) \times_{11} \mu_{NORM(\psi_{12})}(y)),\ (\vartheta_{NORM(\theta_{12})}(x)) \times_{11} \vartheta_{NORM(\rho_{12})}(y))$$

$$x_{21} = (\mu_{NORM(\epsilon_{21})}(x) \times_{11} \mu_{NORM(\psi_{21})}(y)),\ (\vartheta_{NORM(\theta_{21})}(x)) \times_{11} \vartheta_{NORM(\rho_{21})}(y))$$

$$x_{22} = (\mu_{NORM(\epsilon_{22})}(x) \times_{11} \mu_{NORM(\psi_{22})}(y)),\ (\vartheta_{NORM(\theta_{22})}(x)) \times_{11} \vartheta_{NORM(\rho_{22})}(y))$$

By using formula of Cartesian product of "$\times_{11}$" by A$\times_{11}$B=$\{\langle\langle x,y\rangle, \frac{\mu_{NORM(A)}(x) + \mu_{NORM(B)}(y)}{2(sup\,(\mu_{NORM(A)}(x))\,sup\,(\mu_B(y)+1))}, \frac{(\vartheta_{NORM(A)}(x) - inf(\vartheta_{NORM(A)}(x))) + (\vartheta_{NORM(B)}(y) - inf(\vartheta_{NORM(B)}(y)))}{2((1 - inf(\vartheta_{NORM(A)}(x)))\,(1 - inf(\vartheta_{NORM(B)}(y)))+1)}\rangle | \ x \in E_1 \ \&\ y \in E_2\}$.

$$x_{11} = \left( \frac{\mu_{NORM(\epsilon_{11})}(x) + \mu_{NORM(\psi_{11})}(y)}{2(sup\left(\mu_{NORM(\epsilon_{11})}(x)\right)\,sup\left(\mu_{NORM(\psi_{11})}(y)\right)+1)}, \frac{(\vartheta_{NORM(\theta_{11})}(x) - inf\,(\vartheta_{NORM(\theta_{11})}(x))) + (\vartheta_{NORM(\rho_{11})}(y) - inf\,(\vartheta_{NORM(\rho_{11})}(y)))}{2(\left(1 - inf\left(\vartheta_{NORM(\theta_{11})}(x)\right)\right)\left(1 - inf\left(\vartheta_{NORM(\rho_{11})}(y)\right)\right)+1)} \right)$$

$$x_{12} = \left( \frac{\mu_{NORM(\epsilon_{12})}(x) + \mu_{NORM(\psi_{12})}(y)}{2(sup\left(\mu_{NORM(\epsilon_{12})}(x)\right)\,sup\left(\mu_{NORM(\psi_{12})}(y)\right)+1)}, \frac{(\vartheta_{NORM(\theta_{12})}(x) - inf\,(\vartheta_{NORM(\theta_{12})}(x))) + (\vartheta_{NORM(\rho_{12})}(y) - inf\,(\vartheta_{NORM(\rho_{12})}(y)))}{2(\left(1 - inf\left(\vartheta_{NORM(\theta_{12})}(x)\right)\right)\left(1 - inf\left(\vartheta_{NORM(\rho_{12})}(y)\right)\right)+1)} \right)$$

$$x_{21} = \left( \frac{\mu_{NORM(\epsilon_{21})}(x) + \mu_{NORM(\psi_{21})}(y)}{2(\sup\left(\mu_{NORM(\epsilon_{21})}(x)\right) \sup\left(\mu_{NORM(\psi_{21})}(y)\right)+1)}, \frac{(\vartheta_{NORM(\theta_{21})}(x) - \inf(\vartheta_{NORM(\theta_{21})}(x))) + (\vartheta_{NORM(\rho_{21})}(y) - \inf(\vartheta_{NORM(\rho_{21})}(y)))}{2(\left(1-\inf\left(\vartheta_{NORM(\theta_{21})}(x)\right)\right)\left(1-\inf\left(\vartheta_{NORM(\rho_{21})}(y)\right)\right)+1)} \right)$$

$$x_{22} = \left( \frac{\mu_{NORM(\epsilon_{22})}(x) + \mu_{NORM(\psi_{22})}(y)}{2(\sup\left(\mu_{NORM(\epsilon_{22})}(x)\right) \sup\left(\mu_{NORM(\psi_{22})}(y)\right)+1)}, \frac{(\vartheta_{NORM(\theta_{22})}(x) - \inf(\vartheta_{NORM(\theta_{22})}(x))) + (\vartheta_{NORM(\rho_{22})}(y) - \inf(\vartheta_{NORM(\rho_{22})}(y)))}{2(\left(1-\inf\left(\vartheta_{NORM(\theta_{22})}(x)\right)\right)\left(1-\inf\left(\vartheta_{NORM(\rho_{22})}(y)\right)\right)+1)} \right)$$

We have $A \times_{11} B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a normalization of an intuitionistic fuzzy matrices.

## 6. Conclusion:

In this paper, we present an overview of normalized and intuitionistic fuzzy matrices. In this case we discussed ordinary fuzzy matrices into normalization on intuitionistic fuzzy matrices of multi criteria decision making solutions. The cartesian product is utilized to prove several significant results for normalized intuitionistic fuzzy matrices. And Normalization of intuitionistic fuzzy matrices is a vital step in many decision-making processes that involve uncertainty and vagueness. By standardizing the intuitionistic fuzzy values, it ensures that evaluations, aggregations, and grades can be made effectively and meaningfully. This process improves the reliability of decision-making models and makes them more useful in real-world situations, where uncertainty is common.

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